Simple Harmonic Motion, Mass on a Spring, Pendulum & Other Oscillations

A motion that repeats itself over and over is referred to as periodic motion. Any periodic motion may be characterized by its

- **Period** T, which is the time it takes to complete one full cycle. The unit of period is 1 s.
- **Frequency**, which is the ***reciprocal*** of the period. $f \equiv \frac{1}{T}$

Frequency measures the number of cycles per second of the oscillation. The units of f, 1/s is called **Hertz** (Hz).

Simple Harmonic Motion

Simple harmonic motion is motion in which the position of a moving particle is a sinusoidal or cosinusoidal function of time.

$$x(t) = x_m cos(\omega t + \phi)$$

x_m is referred to as the **amplitude** and is the maximum extent of the particle's displacement.

f is the **frequency** or number of oscillations per second.

ϕ is a **phase factor** or **phase angle** in units of radians.

Since the motion is periodic, in time $t = T$ the particle returns to the same position. Thus,

$$x_m cos(\omega t + \phi) = x_m cos[\omega(t+T) + \phi]$$

$$(\omega t + \phi) + 2\pi = \omega(t+T) + \phi$$

$$\omega t + \phi + 2\pi = \omega t + \omega T + \phi$$

$\omega T = 2\pi$ or $\omega = 2\pi / T$

$\omega = 2\pi f$ and is called the **angular frequency** of the particle.

Velocity and Acceleration in Simple Harmonic Motion

We can find the velocity and acceleration of the particle as functions of time by evaluating derivatives of the position function:

$$v(t) = \frac{dx}{dt} = -\omega x_m sin[\omega t + \phi]$$

$$a(t) = \frac{d^2x}{dt^2} = -\omega^2 x_m cos[\omega t + \phi] = -\omega^2 x(t)$$

The **velocity amplitude** is ωx_m .

The **acceleration amplitude** is $\omega^2 x_m$.

Note the following:

- The displacement maxima correspond to velocity minima.
- The displacement maxima correspond to acceleration maxima.
- There is a relative minus sign between *x*(*t*) and *a*(*t*).

The **force** *F* which causes simple harmonic motion is **proportional to** and **opposite to** the **displacement**. Such a force is called a “restoring force.” Consequently, the acceleration is also proportional to and opposite to the displacement,

$$a(t) = \frac{d^2x}{dt^2} = -\omega^2 x(t)$$

The solution of this simple differential equation is $x(t) = x_m cos(\omega t + \phi)$ the sinusoidal function of time given above.

Two important examples of simple harmonic motion are as follows:

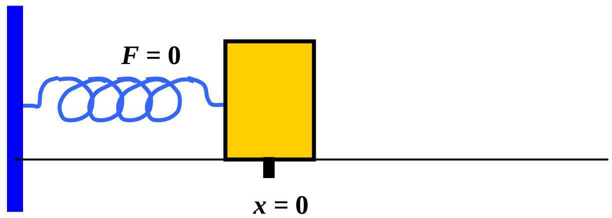
- A mass *m* oscillating freely at the end of a spring of spring constant *k*, for which

$$\omega = \sqrt{\frac{k}{m}}$$

- A mass swinging freely at the end of a pendulum of length *l*, for which

$$\omega = \sqrt{\frac{g}{l}}$$

Mass on a Spring



Consider a mass on a frictionless horizontal surface attached to the end of a spring with a spring constant k. The restoring force obeys Hooke's Law for springs in the elastic region: $F = -kx$.

By Newton's Second Law:

$$F = -kx = ma$$

Because $a = -\omega^2 x$

$$-kx = -m\omega^2 x$$

Solving for ω we get:

$$\omega = \sqrt{\frac{k}{m}}$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$

The equation above shows that stiffer springs (large *k*) give systems smaller periods *T*.

The mechanical energy of a mass on a spring is the sum of the kinetic energy of the mass

$$K(t) = \frac{1}{2}mv^2 = \frac{1}{2}m[-\omega x_m sin(\omega t + \phi)]^2$$

and the potential energy of the spring

$$U(t) = \frac{1}{2}kx^2 = \frac{1}{2}k[x_m cos(\omega t + \phi)]^2$$

Since $\omega = \sqrt{\frac{k}{m}}$

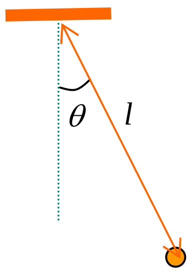
$$E = K(t) + U(t) = \frac{1}{2}k{x_m}^2\left[sin^2(\omega t + \phi) + cos^2(\omega t + \phi)\right] = \frac{1}{2}k{x_m}^2$$

The same result can be obtained from the Law of Conservation of Energy.

Note that the period of the mass-spring system is the same, no matter what the orientation of the system is (horizontal, vertical or inclined).

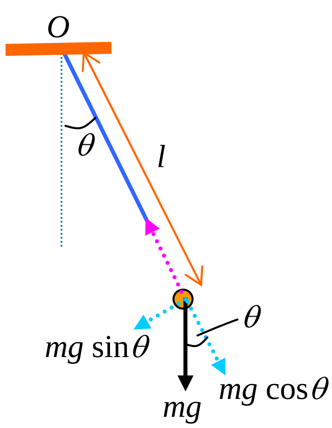
Simple Pendulum

A **simple pendulum** consists of a particle of mass *m*, attached to a frictionless pivot by a thin support wire of length *l*, as shown in the figure.



If the mass is pulled aside by a small angle and released, the pendulum will swing back and forth in simple harmonic motion. It can be shown that the period of this motion is given by the following equation:

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{g/l}} = 2\pi\sqrt{\frac{l}{g}}$$



We can see from FBD that the only force which points along the tangential direction is $-mg\sin\theta$. By considering the torque of the system about *O*, we obtain

$$-lmg\sin(\theta) = I\alpha$$

For small angles $\sin\theta \approx \theta$

$$I\alpha = lmg\theta$$

This is similar to $ma = -kx$, where $m = I$ and $k = Img$. So that by analogy,

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{mgl}{I}}$$

For a point mass on the pendulum $I = ml^2$

$$\omega = \sqrt{\frac{mgl}{ml^2}} = \sqrt{\frac{g}{l}}$$

$$\omega = \sqrt{\frac{g}{l}}$$

since $T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\frac{g}{l}}} = 2\pi\sqrt{\frac{l}{g}}$

$$T = 2\pi\sqrt{\frac{l}{g}}$$

Note that ω and *T* are independent of mass.