OSCILLATIONS AND GRAVITATION

1. SIMPLE HARMONIC MOTION

Simple harmonic motion is any motion that is equivalent to a single component of uniform circular motion. In this situation the velocity is always greatest in the middle of the motion, and the acceleration is greatest at the edges of the motion. In simple harmonic motion, acceleration is always proportional to the negative of displacement. That is

$$a \propto -x$$

We can derive an equation for simple harmonic motion by observing a rotating peg. The *x*-component of the position is simply the radius of the circle, which we will call A, times the cosine of the angle, θ. The angle θ is just the product of the angular velocity ω and time t. Thus, the equation for position as a function of time is

$$x = A\cos\omega t$$

In many situations simple harmonic motion takes place without the involvement of any circular motion. For example, a mass bouncing on a spring executes simple harmonic motion. In these cases, the variable A represents the maximum displacement of the object which is known as the **Amplitude**.

The time it takes to complete one full cycle of the motion is called the **period**, and is written as T. We find the period by increasing the angle $\theta = \omega t$ by 2π and solving for the time. Thus we find

$$T = \frac{2\pi}{\omega}$$

The **frequency**, denoted by f, is the number of complete cycles in a given time interval, and is equal to one over the period:

$$f = \frac{1}{T} = \frac{\omega}{2\pi}$$

Equations for velocity and acceleration can be found either directly, by taking the x-component of the velocity and acceleration in uniform circular motion, or by following the rules of calculus to calculate the velocity and acceleration as derivatives of position. Either way these equations are

$$v = -\omega A \sin \omega t$$

$$a = -\omega^2 A \cos \omega t$$

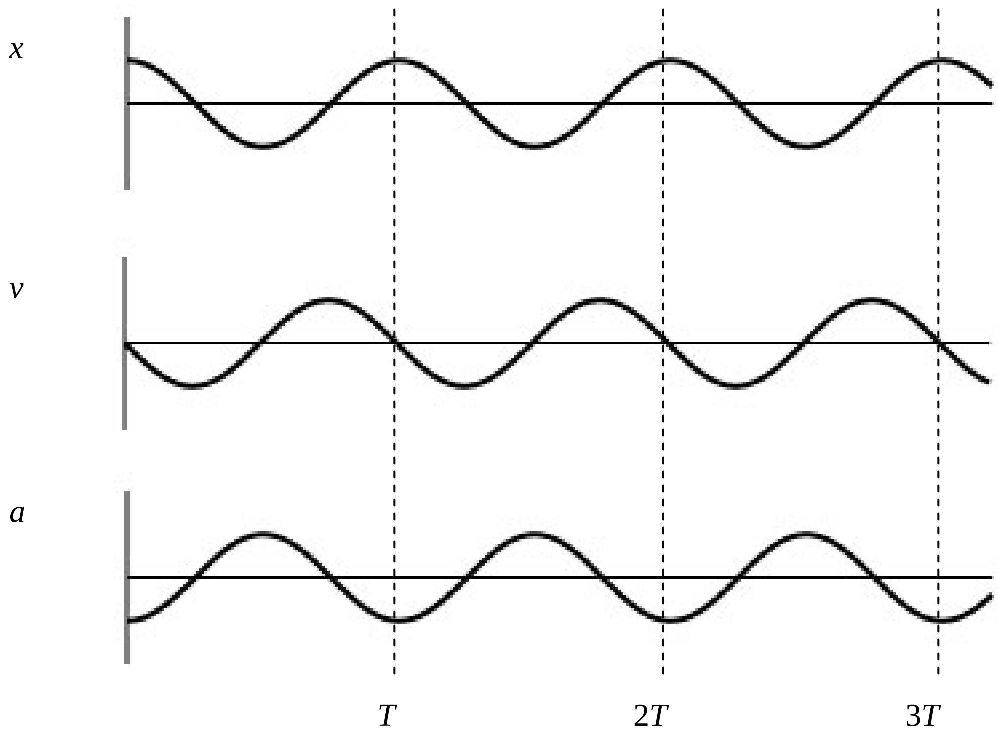
In general the object in simple harmonic motion may not begin this motion with its maximum displacement at time t=0. Including a phase factor ϕ provides for this, resulting in the equations

$$x = A\cos(\omega t + \phi)$$

$$v = -\omega A\sin(\omega t + \phi)$$

$$a = -\omega^2 A\cos(\omega t + \phi)$$

The variables *x, v,* and *a* are graphed below to show how their values compare to one another at different points in the motion.



Notice that, as mentioned before, the acceleration is proportional to the displacement, and the constant of proportionality is -ω^2. In other words

$$a = -\omega^2 x$$

The most common use of this equation is to find the value of ω. We simply compare the equation with the equation for acceleration that we find using the force laws, and read off the value.

2. MASS ON A SPRING

A common example of simple harmonic motion is a **mass on a spring**. The force produced by a spring is governed by **Hooke's Law,** which states

$$F = -kx$$

Here x is the displacement of the spring from **equilibrium**, the point where the spring exerts no force, and k is a constant that indicates how stiff the spring is. We can deduce from this equation that the more a spring is stretched, the greater the force it exerts. The negative sign indicates the spring force is a **restoring force**, that is, when an object subjected to a spring force is displaced from an equilibrium position, the force pushes the object towards that position. The constant k indicates how difficult the spring is to stretch or compress. For stiff springs k is large, for soft springs k is small.

We can use Hooke's Law and Newton's second law ($F = ma$) to study simple harmonic motion both conceptually and quantitatively. First we will consider the conceptual aspects of the motion. Imagine that the spring attached to a mass m lies horizontally on a frictionless table, so that the force of gravity does not affect the motion of the mass. Thus, the force of the spring is the only force that acts on the mass. Moving the mass to the left, results in a large restoring force to the right. Moving the mass to the right, results in a large restoring force directed to the left. Suppose we move the mass to the left and release it. Initially, the restoring force accelerates the mass to the right, but as the mass begins to move, the displacement and thus the force decrease. By the time the mass reaches the equilibrium position, its velocity is at a maximum, but the force - and thus the acceleration - is 0. At this point the inertia of the mass carries it through the equilibrium point to the right. Now the spring exerts a restoring force to the left, which slows the mass. Eventually the mass stops, but by the time it does, it is well to the right of the equilibrium point and it is, once again, subjected to a large restoring force to the left. The process repeats itself, and the object continues to move back and forth, past its equilibrium position.

Next we consider the quantitative aspects of Hooke's law and Newton's second law. By combining the two, we obtain

$$a = -\frac{k}{m}x$$

We can use this to find equations for angular velocity, period, and frequency for a mass on a spring:

$$\omega = \sqrt{\frac{k}{m}}$$

$$T = 2\pi\sqrt{\frac{m}{k}}$$

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

If the spring is vertical and the mass is hanging from it, the description of the motion is somewhat more complicated as the force of gravity is a factor. In this case the force equation becomes

$$kx - mg = ma \qquad *$$

This equation can be simplified to make calculations more straightforward. First consider the position, x_0, where the mass is in equilibrium (i.e. it is not accelerating). Since a=0 at this point, the equation reduces to

$$kx_0 = mg$$

We can now replace mg in the equation (*) above with kx_0 to find

$$F_{\text{net}} = k(x - x_0) = ma$$

The formula is identical to the one for a horizontal spring on a frictionless surface, except that we have shifted our 0 position from the point where the force exerted by the spring is 0 to an equilibrium point where the spring and gravitational forces cancel.

Sometimes it is useful to know the potential energy stored in a spring. As derived earlier the potential energy of a spring is

$$U_{\text{spring}} = \tfrac{1}{2}kx^2$$

The total energy of the mass on the spring system is just the kinetic energy plus the potential energy:

$$E_{\text{tot}} = K + U_{\text{spring}} = \tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2 = \tfrac{1}{2}kA^2$$

We obtained this equation by inputting the values v and x have when the mass is at its maximum displacement ($x = A$, $v = 0$).

3. PENDULUM

Another system with motion that is very nearly simple harmonic is a simple **pendulum**. A pendulum is just a mass at the end of a string. A relatively simple derivation yields an equation for the restoring force:

$$F = -mg\sin\theta$$

θ is the angle the string makes with the horizontal. You may recognize the fact that $-mg\sin\theta$ is the component of the weight of the pendulum, mg, that is tangent to the arc described by the pendulum. Notice that the force is not proportional to the angle of displacement, θ, as required for simple harmonic motion, but instead to the sine of θ. Thus, a pendulum is not a true simple harmonic oscillator, and for large angles a pendulum deviates significantly from simple harmonic motion. However, for small angles, less than about 15°, or .26 radians, the difference between θ expressed in radians and $\sin\theta$ is less than 1%,

$$\theta \approx \sin\theta$$

and thus a pendulum's motion through such angles is very nearly simple harmonic motion, and we can write:

$$F \approx -mg\,\theta = -\frac{mg}{L}\,x$$

where x is the displacement of the pendulum mass and L is the length of the pendulum's string.

If we now use Newton's second law we find

$$a = -\frac{g}{L}\,x$$

From which we can find the angular frequency (ω), the period (T), and frequency (f):

$$\omega = \sqrt{\frac{g}{L}}$$

$$T = 2\pi\sqrt{\frac{L}{g}}$$

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{L}}$$

4. NEWTON'S LAW OF GRAVITATION

Newton postulated that a force exists between any two particles in the universe and that the force follows the relationship:

$$\mathbf{F} = -\frac{Gm_1m_2}{r^2}\hat{\mathbf{r}}$$

That is, the force is proportional to each of the masses, and inversely proportional to the distance between them. If either mass increases the force increases. If the distance increases, the force decreases. The negative sign indicates that the force is attractive and the unit vector $\hat{\mathbf{r}}$ (read as "r hat") shows that the force is along the line between the two masses. Both of these are frequently dropped from the equation when the direction and vector nature of the force are clearly understood.)

This equation is known as **Newton's Law of Gravitation.** Newton used the law to derive the motion of the moon and planets and then showed that his derivations agreed with earlier measurements made by others. Newton's Law of Gravitation describes a force that acts at a distance, and is independent of the medium between the objects.

The constant G, known as the universal gravitational constant, was found experimentally by Henry Cavendish by measuring small gravitational forces that exist between large metal spheres. G has the value

$$G = 6.67\times10^{-11}\,\mathrm{m}^3/\mathrm{kg}\cdot\mathrm{s}^2 = 6.67\times10^{-11}\,\mathrm{N}\cdot\mathrm{m}^2/\mathrm{kg}^2$$

Newton's law applies to individual point masses. For large extended objects, it still applies if we treat an object like all of its mass is concentrated at its center of mass. Thus, we can apply the law to planets or galaxies, or even to people by just treating each object concerned as though its entire mass is concentrated at a single point near their center.

We can also use the concept of **fields** to increase our understanding of gravity. Consider a single particle in space. Without another particle, no gravitational force exists, but we know that if another particle were placed some distance from the first, then a force would be felt by the second particle. We define the gravitational field $\mathbf{g}$ as the force divided by the mass experiencing the force

$$\mathbf{g} = \frac{\mathbf{F}}{m}$$

It is clear from Newton's Second Law that $\mathbf{g}$ is the acceleration of the particle, and in fact that the gravitational field is simply the acceleration due to gravity. Notice that if we insert Newton's Law of gravity for $\mathbf{F}$ we get:

$$\mathbf{g} = \frac{-\dfrac{Gm_1m_2}{r^2}\hat{\mathbf{r}}}{m_2} = -\frac{Gm_1}{r^2}\hat{\mathbf{r}}$$

Since the mass m_2 cancels, the gravitational field at a point at a distance r is independent of the mass of the object subjected to the gravitational force at that point. This is equivalent to Galileo's statement that all freely falling objects experience the same acceleration regardless of mass. Because of this independence of mass, the gravitational field can exist without the presence of a second mass.

We can also define a potential energy due to the gravitational field. We begin by defining gravitational potential energy as being 0 when two objects are infinitely far apart. (This may seem odd, but it is in fact the most convenient convention to use for zero potential energy.) With this convention we find that the potential energy is

$$U_g = -\int_{\infty}^{r} \frac{-Gm_1m_2}{r^2}\,dr = -\frac{Gm_1m_2}{r}$$

Notice that gravitational potential energy is negative. Because particles have less energy when they are close together than when they far apart, the potential energy must be less than 0 (the value when they are infinitely far apart).

Sometimes we want to know the gravitational field and force inside an extended object. In this case the gravitational force depends solely on the matter that is between our point of concern and the center of mass of the surrounding object. Consider, for instance, a spherical shell. From outside the shell the force felt is that of all the mass as if it were at the center of mass of the shell. However, from inside the shell we feel no force at all. If we drill into the earth the gravitational force becomes weaker and weaker, because increasingly less mass is between the earth's center and us. When we reach the center the force is 0 because no mass lies closer than us to the center of mass.

5. ORBITS OF PLANETS AND SATELLITES

People have studied the motion of the planets for thousands of years. Newton's Law of Gravitation is a powerful tool for describing this motion. Consider a planet moving in a circular orbit around the sun: gravity provides the centripetal force and we can set the equations for the forces equal to find equations for the centripetal acceleration, velocity, and period of a planet.

$$\frac{GM_sM_p}{r^2} = M_pa_c$$

$$a_c = \frac{GM_s}{r^2}$$

$$a_c = \frac{v^2}{r}$$

$$v = \sqrt{\frac{GM_s}{r}}$$

$$v = \frac{2\pi r}{T}$$

$$T^2 = \left(\frac{4\pi^2}{GM_s}\right) r^3$$

Notice that the centripetal acceleration, velocity, and period of the planet are all independent of the planet's mass. Thus, if a tiny satellite were at the same distance from the sun as the earth, it would follow the same orbit as the earth, at the same speed, and in the same time. The equations above can also be used for moons or satellites orbiting the earth or other planets by replacing the mass of the sun with the mass of the planet being orbited.

The energy of an object in orbit can also be determined from what we have learned. Let m be the mass of the satellite, and M be the mass of the sun or planet being orbited. We can write the total energy (kinetic plus potential) as

$$E = K + U_g = \tfrac{1}{2} mv^2 - \frac{GMm}{r}$$

To simplify this for a circular orbit, we set the gravitational force equal to the centripetal force, as above. Alternatively, we may simply substitute in our equation we determined above for *v*. Using either approach yields this equation for the total energy:

$$E = -\frac{GMm}{2r}$$

This is just half the potential energy at this point. (The kinetic energy is positive and half as big as the potential energy so in effect the two partially cancel.) The energy of the orbiting body is negative because the satellite is bound to the sun or to the planet it is orbiting. That is it has less energy than if the two objects were infinitely far apart and at rest. As a result, the satellite can never escape to infinity.

For a satellite (a planet orbiting the sun or a moon orbiting a planet) that is following an elliptical orbit the equation for its energy is reached by replacing the radius by the average radius R or by the length of the semi-major axis a:

$$E = -\frac{GMm}{2a}$$

Suppose we want to determine how much energy is required for a rocket to escape from a planet. Once the rocket is infinitely far away, its potential energy will be 0. At that point it will have escaped, and once it has escaped, its velocity may be 0. Therefore, the rockets total energy must be at least 0 to escape. (If the total energy is negative the rocket is still bound to the planet.) Setting this equal to the initial energy we find

$$E = K + U_g = \tfrac{1}{2} m v_i^2 - \frac{GMm}{r_i} = 0$$

$$K = -U_g$$

$$v = \sqrt{\frac{2GM}{r_i}}$$

The subscript i indicates the initial values of v and r. For a rocket escaping from the earth, the initial radius is just the radius of the earth, $r_i = R_e$. Notice that it takes twice as much kinetic energy to escape from an orbit with a given radius than to orbit the planet at that radius. Also the escape velocity from an orbit of a given radius is equal to the square root of 2 times the orbital velocity at that radius.

$$K_{\text{esc}} = 2K_{\text{orb}}$$

$$v_{\text{esc}} = \sqrt{2}\, v_{\text{orb}}$$

We can use conservation of angular momentum to learn a great deal about the velocity of a planet (or of any other satellite) in an elliptical orbit. Because the force of gravity is radial (that is, it acts along the line joining the planet and the sun) it exerts no torque, and therefore angular momentum is conserved. We can write this as

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = \mathrm{m}\,\mathbf{r} \times \mathbf{v} = \text{constant}$$

$$L = mvr \sin\theta = \text{constant}$$

Because angular momentum is conserved each planet orbits in a single plane. The principle can also be used to relate the velocity at various point. Consider aphelion (the orbit's furthest point from the sun) and perihelion (the orbit's closest point to the sun). At these two points the angle between the radius and momentum is 90°, and thus $\sin\theta = 1$. With this value, conservation of angular momentum can be used to relate the velocities at these two points.

$$v_{\text{a}} r_{\text{a}} = v_{\text{p}} r_{\text{p}}$$

$$v_{\text{p}} = \frac{r_{\text{a}}}{r_{\text{p}}} v_{\text{a}}$$

Notice the velocity is greater at the perihelion.

The same idea can be used to find the velocity at any point relative to the velocity at the perihelion:

$$vr \sin\theta = v_p r_p$$

$$v = \frac{v_p r_p}{r \sin\theta}$$