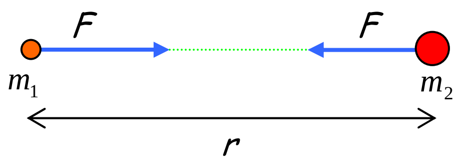
# Newton's Law of Gravity and
# Orbits of Planets & Satellites

## The Universal Law of Gravitation

Newton's Universal Law of Gravitation states that any two point masses attract each other with a force proportional to the product of their masses and inversely proportional to the square of the distance between them. The force of universal gravitation is a very weak force and only becomes noticeable when at least one of the objects is extremely massive. The force acts equally on both masses.




$F = G\frac{m_1 m_2}{r^2}$ where $F$ = gravitational force on each mass (N)

Where $G = 6.67 \times 10^{-11}$ N m$^2$/kg$^2$ is universal gravitational constant, $m_1$, $m_2$ are the point masses, $r$ is the distance between the masses.

The direction of this force is always along the line between the two masses. If one object is spherically symmetric, the force exerted on another object outside the sphere is exactly as if the whole spherical mass were concentrated at the center.

## Gravitational Field

A gravitational field is the force field created around an object that causes gravitational attraction of other objects. The gravitational field strength at a point in a gravitational field is the force per unit of mass of an object placed at that point. From Newton's Second Law of Motion, we can regard "force per unit mass" as being equivalent to **acceleration**.

Therefore, gravitational field strength is another name for **acceleration due to gravity**.

$$g = \frac{F}{m} = \frac{G\frac{mM_E}{r^2}}{m} = G\frac{M_E}{r^2},$$

where $r$ is the distance of the test mass $m$ from the center of the mass $M$ creating gravitational field. The unit of gravitational field strength is the same as that of acceleration, i.e. N/kg, or m/s$^2$.

For the gravitational field created by Earth, when $r < R_E$, the gravitational field $g$ is proportional to the distance of the test mass from the center of Earth. When $r \approx R_E$, the gravitational field $g$ = 9.81m/s$^2$. When $r > R_E$, the curve is an inverse square law.

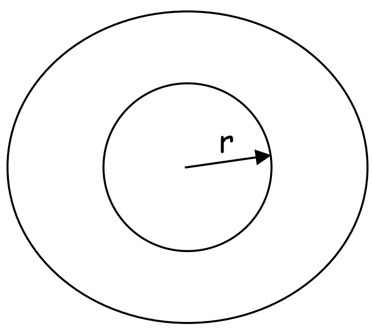
**Shell Theorem**

The shell theorem states the following:

1. A uniform spherical shell of matter attracts a particle that is outside as if all the shell's mass were concentrated at its center.

2. A uniform shell of matter exerts no net gravitational force on a particle located inside the shell.

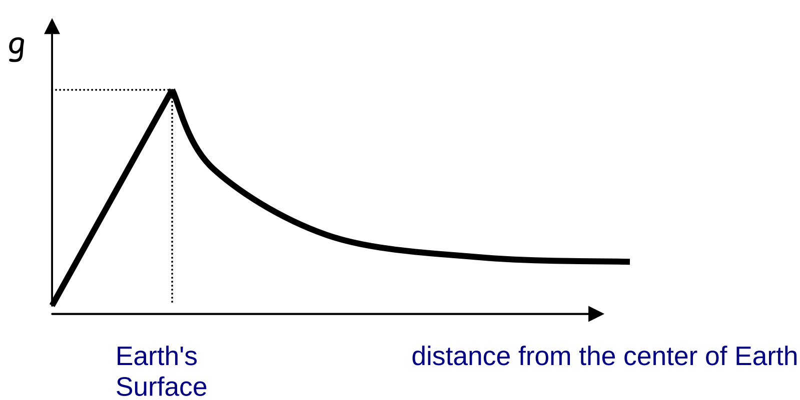
Applying the shell theorem to the gravitational field created by Erath, we can see that for $r < R_E$ (points inside the Earth), the gravitational field strength is directly proportional to the distance from the center of Earth $r$. By the shell theorem, the masses outside the test mass give no force




to the test mass, and the gravitational force is solely the result of the remaining mass $M = V \cdot \rho = \frac{4}{3}\pi r^3 \rho$ (remember that the volume of a sphere is $\frac{4}{3}\pi r^3$) where $\rho$ is the average density of the Earth. However, the mass $M$ can be regarded as a point mass located at the center of Earth, hence can express the gravitational field as

$$g = \frac{GM}{r^2} = \frac{G(\frac{4}{3}\pi r^3 \rho)}{r^2} = \frac{4}{3}G\pi r \rho = \frac{4}{3}G\pi r(\frac{M_E}{\frac{4}{3}\pi R_E^3}) = \frac{GM_E}{R_E^3} r \ .$$

As shown in the graph below, when $r < R_E$, the gravitational field $g$ is proportional to the distance of the test mass and the center of earth. When $r > R_E$, the curve is an inverse square law.



**Gravitational Potential Energy**

The gravitational potential energy of a system consisting of masses *m* and *M* with a distance *r* between their centers is

$$U = -G\frac{M\ m}{r}$$

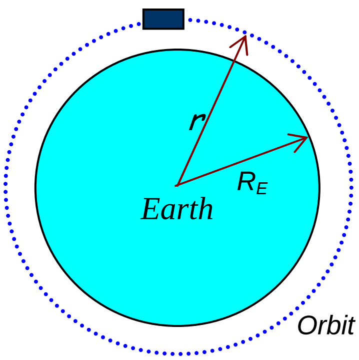
The negative sign reflects on the fact that the gravitational force is attractive. *U* is equal to the work done of the gravitational force to move a mass *m* from a point *r* measured from the center of the mass *M* to infinity. Note that *U* approaches zero as *r* approaches infinity.

**Satellite Orbits**

The centripetal force for a satellite in its orbit is the gravitational attraction. Since the gravity can be considered the only force acting on the satellite, by Newton's Second Law:

$F_g = ma_c$ where $a_c = \frac{v^2}{r}$ and $F_g = \frac{GM_E m}{r^2}$. Thus $\frac{mv^2}{r} = \frac{GM_E m}{r^2}$

$v^2 = \frac{GM_E}{r}$ Multiplying both the numerator and denominator of this expression by $R_E^2$ and rearranging, we get $v^2 = \frac{GM_E}{r} = \frac{GM_E}{R_E^2}\frac{R_E^2}{r} = g\frac{R_E^2}{r}$




For example, if the orbit of the satellite is 200 km from the Earth's surface, and considering that the Earth's radius is 6400 km, the speed of the satellite can be found as

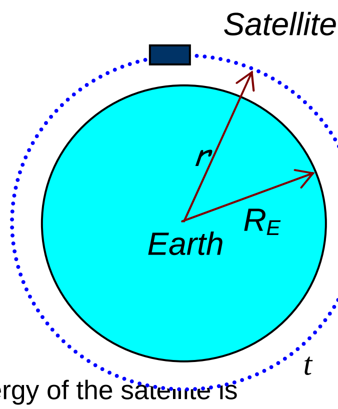
$$
\begin{aligned}
v^2 &= (9.81\ \text{m/s}^2)\left(\frac{R_E^2}{r}\right) \\
&= (9.81\ \text{m/s}^2)\frac{(6.40\times10^6\,\text{m})^2}{(6400+200)\times10^3\,\text{m}} \\
&= (9.81\ \text{m/s}^2)\frac{(6.40\times10^6\ \text{m})^2}{6.60\times10^6\ \text{m}} \\
v^2 &= 6.09\times10^7\,\text{m}^2/\text{s}^2 \\
v &= 7.80\times10^3\ \text{m/s}
\end{aligned}
$$

The time for the satellite to make one complete orbit of the earth is

$$T = \frac{2\pi r}{v}\ ;\ T \approx 89\,\text{minutes}$$

**Mechanical Energy of a Satellite**

Consider a satellite moving along a circular orbit with tangential velocity *v* at a distance *r* from the center of Earth.




The potential energy of the satellite is

$$U(r) = -\frac{GM_E m}{r}.$$

The total mechanical energy = *K* + *U*

Hence, we have

$$E = \frac{1}{2}mv^2 - \frac{GM_E m}{r}.$$

But the centripetal force is provided by the gravitational attraction

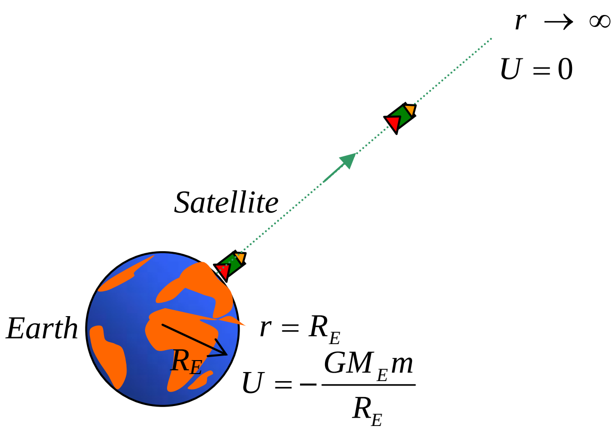
$$\frac{mv^2}{r} = \frac{GM_E m}{r^2}$$

which gives the velocity *v*, i.e. $v^2 = \frac{GM_E}{r}$. Thus, we obtain the total mechanical energy

$$E = \frac{1}{2}m(\frac{GM_E}{r}) - \frac{GM_E m}{r} = -\frac{GM_E m}{2r}$$

**Launching a Satellite**

The satellite can escape from the Earth whenever it has enough energy. “Escape” in this context means that the satellite gets to the point where the gravitational force is approaching zero. The potential energy of the satellite at this point is zero as well.




As it leaves the surface of Earth, the potential energy is $U = -\frac{GM_E m}{R_E}$ and its kinetic energy is $K = \frac{1}{2}mv^2$. Then, the minimum speed for the satellite to escape can be found from the Law of Conservation of Energy: $\frac{1}{2}mv^2 - \frac{GM_E m}{R_E} = 0$. (We assume that the kinetic energy of the satellite at the distant point is zero as well.)

$$\frac{1}{2}mv^2 = \frac{GM_E m}{R_E}$$

$$v = \sqrt{\frac{2GM_E}{R_E}} = \sqrt{2(\frac{GM_E}{R_E^2})R_E} = \sqrt{2gR_E}$$

$$= \sqrt{2(9.81\,\text{m/s}^2)(6.4\times10^6\text{ m})} \approx 11,000\,\text{m/s} = 11\text{km/s}$$

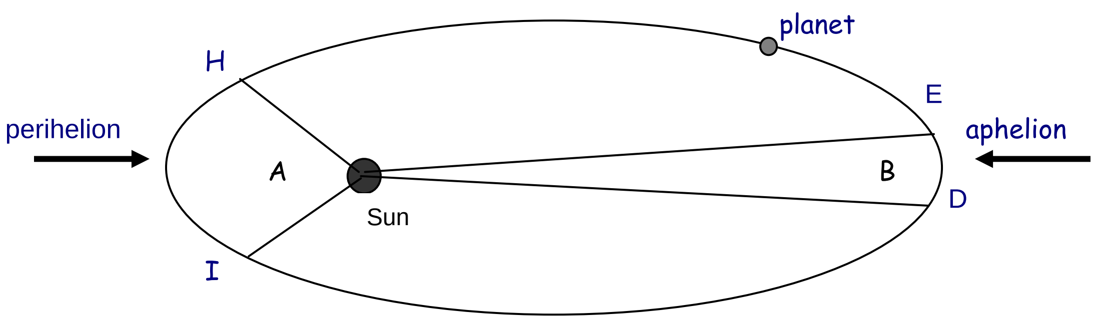
**Kepler's Laws**

In 1609, Kepler, a German astronomer, published his three laws on the motion of planets:

1. The planets move in elliptical orbits with the sun at one focus.

2. The line connecting a planet to the sun sweeps out equal areas in equal times.

3. For every planet, the ratio of the cube of the average orbital radius to the square of the period of revolution is the same, i.e. $\frac{R^3}{T^2}$ = constant.




As area *A* = area *B*, then the time it takes to move from D to E is the same as to move from H to I. The constant for Kepler's third law comes from Newton's second law, $F = ma$.

$$\frac{GM_s M_p}{R^2} = M_p a_c$$
$$\frac{GM_s M_p}{R^2} = M_p \frac{v^2}{R}$$
$$GM_s = v^2 R$$

Since $v = \frac{2\pi R}{T}$

$$GM_s = \frac{4\pi^2 R^2}{T^2} R$$
$$\frac{R^3}{T^2} = \frac{GM_s}{4\pi^2}$$

That is, for a given central body of mass *M*, the ratio $\frac{R^3}{T^2}$ is constant and equal to $\frac{GM}{4\pi^2}$ for all its satellites.